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Extension of the Prandtl–Batchelor theorem to three-dimensional flows slowly varying in one direction

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According to the Prandtl–Batchelor theorem for a steady two-dimensional flow with closed streamlines in the inviscid limit the vorticity becomes constant in the region of closed streamlines. This is not true for three-dimensional flows. However, if the variation of the flow field along one direction is slow then it is possible to expand the solution in terms of a small parameter characterizing the rate of variation of the flow field in that direction. Then in the leading-order approximation the projections of the streamlines onto planes perpendicular to that direction can be closed. Under these circumstances the extension of the Prandtl–Batchelor theorem is obtained. The resulting equations turned out to be a three-dimensional analogue of the equations of the quasi-cylindrical approximation.

1. Introduction

It is well known that two-dimensional Euler equations reduce (by introducing the streamfunction ψ) to the Poisson equation with the vorticity depending only on ψ as the source term. When solving Euler equations in a domain containing regions of closed streamlines, the solution possesses a degree of non-uniqueness. This non-uniqueness is due to the fact that the boundary conditions imposed at infinity do not determine the vorticity on the streamlines that do not originate at infinity, as it is the case in the region of closed streamlines. However, if viscous effects are considered, this problem is overcome. Prandtl (1905) noticed and Batchelor (1956) proved that as viscosity tends to zero, the vorticity tends to a constant value in regions of closed streamlines. Batchelor (1956) also extended this analysis to axisymmetric flows, and established the value of the constant vorticity for the case of a boundary with circular geometry. The well-known Prandtl–Batchelor theorem relies on the flow having closed streamlines, a requirement that a general recirculating three-dimensional flow at large Reynolds number does not necessarily possess. Hence extending the Prandtl–Batchelor theory to the three-dimensional case is difficult. Only when certain symmetries are imposed may the closed streamline theory still be applicable. For example, by assuming that the velocity field is independent of the axial direction z and imposing a constant axial pressure gradient, Blennerhassett (1979) obtained an integral condition that the axial velocity satisfies. Grimshaw (1968) considered a three-dimensional flow with nested closed stream surfaces and derived an integral condition for the vorticity under such assumption.

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Childress, Landman & Strauss (1989), extended the Prandtl–Batchelor results to flows with helical symmetry, recovering as a particular case Blennerhassett's results. More recently Mezic (2002) extended the Prandtl–Batchelor theory to steady three-dimensional flows in a bounded domain in the case when the streamlines do not cross the boundary domain. By time averaging the Navier–Stokes equations along the path of a material particle he obtained two conditions that the velocity and vorticity vector satisfy. These conditions reduce to those of the Prandtl–Batchelor theorem when assuming closed paths.

Concerning two-dimensional flows, the Prandtl–Batchelor theory was extended to compressible flows by Neiland (1970) and Neiland & Sychev (1970), to temperature field by Chernyshenko (1983*a*), to spatially periodic flows by Chernyshenko (1983*b*) and to stratified flows by Kamachi, Saitou & Honji (1985). The work of Buldakov, Chernyshenko & Ruban (2000) on flows with suction should especially be mentioned here as it considered a case of non-closed streamlines. This work used asymptotic techniques and expanded the velocity field in terms of a small parameter so that the leading-term flow pattern contained closed streamlines. Choosing these trajectories as the integration contour made it possible to calculate the vorticity distribution.

The purpose of this paper is to extend the Prandtl–Batchelor theory to three-dimensional flows slowly varying in one direction. This case possesses a degree of non-uniqueness in exactly the same way as the two-dimensional Euler equations do in regions with closed streamlines. Thus the extension of the Prandtl–Batchelor theorem provides extra information in order to reduce the degree of uncertainty in the velocity distribution. While taking into account three-dimensionality in a conveniently compact way, this extension turned out to be an analogue of the quasi-cylindrical equations used for describing behaviour of streamwise vortices, rotating jets, vortex-breakdown phenomenon and some other problems. Hence, the equations obtained below might be used for studying similar phenomena in non-axisymmetric cases, like, for example, a streamwise vortex in the vicinity of a wall, as well as for problems involving flows in which slow variation in one direction is due to the geometry of the boundary.

2. Problem formulation

While the theorem we are going to derive is general, for the sake of clarity we will derive it in the context of the following boundary-value problem. Consider the steady incompressible Navier–Stokes equations in Cartesian coordinates (xyz) under the transformation $\widetilde{z}=\varepsilon z$ for the velocity $\boldsymbol{u}=(u, v, w)=\boldsymbol{u}(x, y, \widetilde{z})$ and pressure $p=p(x, y, \widetilde{z})$:

$$\left.\begin{aligned}
u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\varepsilon w\frac{\partial u}{\partial \widetilde{z}}+\frac{\partial p}{\partial x}&=\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\varepsilon^2\frac{\partial^2 u}{\partial \widetilde{z}^2}\right),\\
u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\varepsilon w\frac{\partial v}{\partial \widetilde{z}}+\frac{\partial p}{\partial y}&=\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\varepsilon^2\frac{\partial^2 v}{\partial \widetilde{z}^2}\right),\\
u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+\varepsilon w\frac{\partial w}{\partial \widetilde{z}}+\varepsilon\frac{\partial p}{\partial \widetilde{z}}&=\frac{1}{Re}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\varepsilon^2\frac{\partial^2 w}{\partial \widetilde{z}^2}\right),\\
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\varepsilon\frac{\partial w}{\partial \widetilde{z}}=0,\quad \boldsymbol{u}|_\sigma&=\boldsymbol{u}_w(x, y, \widetilde{z}),\quad \int_{S_{max}} w\,\mathrm{d}S=q,
\end{aligned}\right\}\tag{2.1}$$

where u, v, w are the Cartesian velocity components, Re is the Reynolds number, q is the flow rate and σ is the flow domain boundary defined by $f(x, y, \varepsilon z)=0$. System (2.1) is made dimensionless by scaling all variables with the characteristic

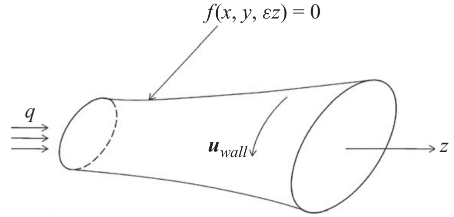

FIGURE 1. Flow domain $f(x, y, \varepsilon z) = 0$.

values for the transverse flow, so that the defined Reynolds number depends on the characteristic transverse velocity and on the characteristic transverse length that is the characteristic scales for the motion in the plane $\tilde{z} = \text{constant}$. We assume that the boundary continues infinitely in the z direction, and that the domain has a cross-section S_{max} that is closed in planes $\tilde{z} = \text{constant}$ and varying with $\tilde{z}$ (see figure 1). The boundary is implied to be impermeable, but a non-zero (in general) tangential velocity $\boldsymbol{u}_w$ is imposed on it. The problem is to derive the closed set of governing equations and boundary conditions for the flow in the limit $Re \to \infty$, $\varepsilon \to 0$, assuming that in this limit the in-plane (plane perpendicular to z-axis) components of the flow velocity form only one nested set of closed contours which are the projections of the limiting streamlines onto that plane.

This problem is non-trivial for the following reason. As $Re \to \infty$ and $\varepsilon \to 0$ with $\tilde{z}$ fixed equations (2.1) tend to the two-dimensional Euler equations with three velocity components, which can be reduced to

$$\left.\begin{aligned} &\nabla^2_{(2)}\psi + \Omega(\psi, \tilde{z}) = 0, \quad H'_\psi + \Omega - WW'_\psi = 0, \\ &\psi = \psi_w(\tilde{z}) \text{ on } f(x, y, \tilde{z}) = 0, \quad \int_{S_{max}} W \, \mathrm{d}S = q, \end{aligned}\right\} \tag{2.2}$$

where $\omega = \partial v/\partial x - \partial u/\partial y$ is the vorticity, $\Omega(\psi, \tilde{z}) \equiv \lim_{\varepsilon\to 0, Re\to\infty} \omega$, $W(\psi, \tilde{z}) \equiv \lim_{\varepsilon\to 0, Re\to\infty} w$ and $H \equiv \boldsymbol{u} \cdot \boldsymbol{u}/2 + p$ are the limiting values of the axial vorticity, axial velocity and Bernoulli function respectively, while $\psi(x, y, \tilde{z})$ is the streamfunction defined by $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. Appropriate boundary conditions are also included. Observe the notation $\nabla_{(2)} = (\partial/\partial x, \partial/\partial y, 0)$ whereas the prime and subscript ψ denotes differentiation with respect to the streamfunction at constant $\tilde{z}$. Clearly, the solution of (2.2) (which represents a three-dimensional inviscid flow slowly varying in the z direction) is not unique, since W and Ω are arbitrary functions of ψ for each $\tilde{z} = \text{constant}$. Since we assume that the contours $\psi = \text{constant}, \tilde{z} = \text{constant}$ are closed, $\Omega(\psi, \tilde{z})$ and $W(\psi, \tilde{z})$ cannot be found from boundary conditions. Therefore the problem is to find extra conditions for W and Ω that remain valid in the limit, hence obtaining extra information that determins the velocity field of the Euler flow in each plane $\tilde{z} = \text{constant}$.

3. Solution by the Prandtl–Batchelor method

3.1. *Derivation of extra conditions for W and Ω*

In general, the extra conditions required here can be obtained as solvability conditions for higher-order terms of the expansion of the solution. While we checked that this approach would give the same result, its rigorous presentation would be very involved, since, due to the formation of a boundary layer near the domain boundary, the higher-order term in question is the third term of the expansion. Instead, we will follow the

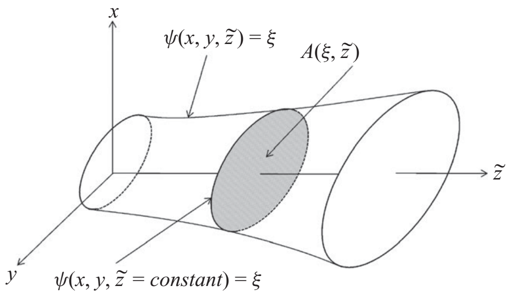

FIGURE 2. Surface generated by the streamfunction ψ and the cross-section area A.

idea of Batchelor, and derive an integral condition (that is a condition on an integral of the solution) that the solution satisfies at arbitrary Re and ε and that remains non-trivial in the limit.

Introducing vorticity $\boldsymbol{\omega} = (\zeta, \chi, \omega)$ and denoting $\varepsilon Re = k$ one obtains from (2.1) the equations for axial components of velocity and vorticity which can be written in a conservative form as

$$\nabla_{(2)} \cdot \boldsymbol{u}_{(2)} + \varepsilon \frac{\partial w}{\partial \widetilde{z}} = 0, \tag{3.1}$$

$$\nabla_{(2)} \cdot \left(\boldsymbol{u}_{(2)}\omega - \boldsymbol{\omega}_{(2)} w - \frac{\varepsilon}{k} \nabla_{(2)}\omega \right) - \frac{\varepsilon^3}{k} \frac{\partial^2 \omega}{\partial \widetilde{z}^2} = 0, \tag{3.2}$$

$$\nabla_{(2)} \cdot \left(w\boldsymbol{u}_{(2)} - \frac{\varepsilon}{k} \nabla_{(2)} w \right) + \varepsilon \frac{\partial w^2}{\partial \widetilde{z}} = -\varepsilon \frac{\partial p}{\partial \widetilde{z}} + \frac{\varepsilon^3}{k} \frac{\partial^2 w}{\partial \widetilde{z}^2}. \tag{3.3}$$

Here, a subscript (2) denotes a projection on the plane $\widetilde{z} = \text{constant}$, so that $\boldsymbol{X}_{(2)} = (X_1, X_2, 0)$. We will now assume that $Re \to \infty$ and $\varepsilon \to 0$ in such a way that $k = \text{constant}$. In this case viscous effects and three-dimensional effects turn out to be of the same order, thus resulting in the most general (distinguished) limit. The form of the above equations suggests the use of Green's theorem. For any vector $\boldsymbol{F}_{(2)}$ it states that

$\int_S \nabla_{(2)} \cdot \boldsymbol{F}_{(2)}\, \mathrm{d}S = \int_C \boldsymbol{F}_{(2)} \cdot \widehat{n}\, \mathrm{d}s$, where C is an oriented closed contour with external normal $\widehat{n}$, S is the region enclosed by C, and s is the arclength. Consider a closed contour C located in the plane $\widetilde{z} = \text{constant}$. Applying Green's theorem to (3.2) and (3.3) and dividing by ε we obtain the integral conditions:

$$\int_C \left(\frac{\boldsymbol{u}_{(2)}\omega - \boldsymbol{\omega}_{(2)} w}{\varepsilon} - \frac{1}{k} \nabla_{(2)}\omega \right) \cdot \widehat{n}\, \mathrm{d}s = O(\varepsilon^2), \tag{3.4}$$

$$\int_C \left(\frac{w\boldsymbol{u}_{(2)}}{\varepsilon} - \frac{1}{k} \nabla_{(2)} w \right) \cdot \widehat{n}\, \mathrm{d}s = -\int_S \frac{\partial}{\partial \widetilde{z}} \left(p + w^2 \right) \mathrm{d}S + O(\varepsilon^2). \tag{3.5}$$

We now choose the contour C to coincide with an (assumed closed) streamline of the in-plane part of the leading-order term of the solution. This streamline lies in the surface $\psi(x, y, \widetilde{z}) = \xi$, $\xi = \text{constant}$, and encloses region S in the plane $\widetilde{z} = \text{constant}$, the area of which will be denoted A (see figure 2). As discussed in § 2, as $\varepsilon \to 0$ the axial vorticity and velocity tend to $\Omega = \Omega(\psi, \widetilde{z})$ and $W = W(\psi, \widetilde{z})$. Hence, from (3.4) it follows that

$$\lim_{\varepsilon \to 0} \int_C \frac{\boldsymbol{u}_{(2)}\omega}{\varepsilon} \cdot \widehat{n}\, \mathrm{d}s = \Omega \lim_{\varepsilon \to 0} \left(\frac{1}{\varepsilon} \int_C \boldsymbol{u}_{(2)} \cdot \widehat{n}\, \mathrm{d}s \right) = -\Omega \int_S \frac{\partial W}{\partial \widetilde{z}}\, \mathrm{d}S, \tag{3.6}$$

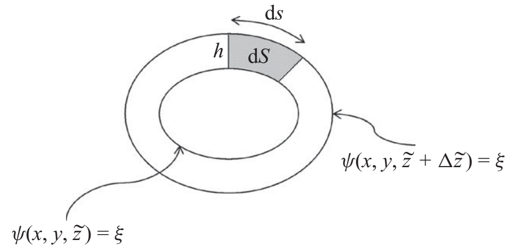

FIGURE 3. Superposition of the contours $\psi(x, y, \widetilde{z} + \Delta\widetilde{z}) = \xi$ and $\psi(x, y, \widetilde{z}) = \xi$ separated by the perpendicular distance h.

where we factored Ω from the integral since it is constant along the contour of integration and used continuity equation in the form $\int_C \boldsymbol{u}_{(2)} \cdot \widehat{n}\, \mathrm{d}s = -\varepsilon \int_S \partial w/\partial\widetilde{z}\, \mathrm{d}S$. Similarly,

$$\lim_{\varepsilon \to 0} \int_C \frac{\boldsymbol{\omega}_{(2)} w}{\varepsilon} \cdot \widehat{n}\, \mathrm{d}s = W \lim_{\varepsilon \to 0} \left(\frac{1}{\varepsilon} \int_C \boldsymbol{\omega}_{(2)} \cdot \widehat{n}\, \mathrm{d}s \right) = -W \int_S \frac{\partial \Omega}{\partial \widetilde{z}}\, \mathrm{d}S, \tag{3.7}$$

where the solenoidal property of the vorticity was used in the form $\int_C \boldsymbol{\omega}_{(2)} \cdot \widehat{n}\, \mathrm{d}s = -\varepsilon \int_S \partial\omega/\partial\widetilde{z}\, \mathrm{d}S$. Finally,

$$\lim_{\varepsilon \to 0} \int_C \nabla_{(2)}\omega \cdot \widehat{n}\, \mathrm{d}s = \int_C \nabla_{(2)}\Omega \cdot \widehat{n}\, \mathrm{d}s = \Omega'_\xi \int_C \nabla_{(2)}\psi \cdot \widehat{n}\, \mathrm{d}s = -\Omega'_\xi \Gamma, \tag{3.8}$$

where the relation $\nabla_{(2)}\Omega = \Omega'_\psi \nabla_{(2)}\psi$, the fact that Ω'_ψ can be factored out of the integral, and the definition of circulation $\Gamma = -\int_C \nabla_{(2)}\psi \cdot \widehat{n}\, \mathrm{d}s$ were employed. Note that Ω'_ψ represents the partial derivative of $\Omega(\psi, \widetilde{z})$ with respect to ψ at constant $\widetilde{z}$, whereas Ω'_ξ denotes the corresponding derivative evaluated at $\psi = \xi$. Results similar to (3.6) and (3.8) can also be obtained for (3.5). Thus by using formulas (3.6)–(3.8) and similar results for (3.5) one may conclude that as $\varepsilon \to 0$, $k =$ constant equations (3.4) and (3.5) tend to

$$-\Omega \int_S \frac{\partial W}{\partial \widetilde{z}}\, \mathrm{d}S + W \int_S \frac{\partial \Omega}{\partial \widetilde{z}}\, \mathrm{d}S + \frac{1}{k}\Omega'_\xi \Gamma = 0, \tag{3.9}$$

$$W \int_S \frac{\partial W}{\partial \widetilde{z}}\, \mathrm{d}S - \frac{1}{k} W'_\xi \Gamma = \int_S \frac{\partial p}{\partial \widetilde{z}}\, \mathrm{d}S + \int_S \frac{\partial W^2}{\partial \widetilde{z}}\, \mathrm{d}S \tag{3.10}$$

with $\Gamma = \int_S \Omega\, \mathrm{d}S$. Relationships (3.9) and (3.10) are the extra conditions which W and Ω should satisfy. Note that for a purely two-dimensional flow (3.9) reduces to the well-known Prandtl–Batchelor theorem stating that $\Omega'_\xi = 0$.

3.2. *Introduction of the Bernoulli function H*

It is convenient to introduce the Bernoulli function H into (3.10) in order to reduce the number of dependent variables. Consider the infinitesimal surface element $\mathrm{d}S = h(s)\, \mathrm{d}s$ (figure 3), where s is the arclength and $h(s)$ is the contour displacement from $\psi = \xi$ to $\psi = \xi + \Delta\xi$, $h \approx \Delta\xi/(\partial\psi/\partial n)$ where $\nabla_{(2)}\psi \cdot \widehat{n} \equiv \partial\psi/\partial n$. Then, substituting the Bernoulli function $H = u^2/2 + p + W^2/2$, $u = |\boldsymbol{u}_{(2)}|$ into (3.10) and differentiating the resulting equation with respect to ξ one obtains

$$W'_\xi \int_S \frac{\partial W}{\partial \widetilde{z}}\, \mathrm{d}S = \int_C \frac{\partial H}{\partial \widetilde{z}} \frac{\mathrm{d}s}{\partial\psi/\partial n} \pm \int_C \frac{\partial u}{\partial \widetilde{z}}\, \mathrm{d}s + \frac{1}{k} \left[W'_\xi \Gamma \right]'_\xi,$$

where signs $\pm$ correspond to counterclockwise and clockwise rotation of the flow, respectively. Applying Stokes theorem to $\pm\int_C(\partial u/\partial\tilde{z})\,\mathrm{d}s$, the latter equation can be rewritten as

$$W'_\xi\int_S\frac{\partial W}{\partial\tilde{z}}\,\mathrm{d}S-\int_S\frac{\partial\Omega}{\partial\tilde{z}}\,\mathrm{d}S=\int_C\frac{\partial H}{\partial\tilde{z}}\frac{\mathrm{d}s}{\partial\psi/\partial n}+\frac{1}{k}\left[W'_\xi\Gamma\right]'_\xi. \tag{3.11}$$

3.3. *Some useful relations involving the cross-section area A*

In §3, A was defined as the area of the cross-section that is enclosed by the contour $\psi=\xi$, $\tilde{z}=$constant, i.e. $A(\xi,\tilde{z})=\int_{S(\xi,\tilde{z})}\mathrm{d}S$. Using this function, let us calculate the partial derivative of A with respect to ξ at constant $\tilde{z}$. From the definition of derivative, $A'_\xi=\lim_{\Delta\xi\to0}(1/\Delta\xi)\int_{S(\xi+\Delta\xi,\tilde{z})-S(\xi,\tilde{z})}\mathrm{d}S$. In order to find this limit we substitute the infinitesimal surface element as $\mathrm{d}S=h(s)\,\mathrm{d}s$, $h=\Delta\xi/(\partial\psi/\partial n)$ and take the limit. This reduces the area integral to the integral over a curve:

$$A'_\xi=\int_C\frac{\mathrm{d}s}{\partial\psi/\partial n}. \tag{3.12}$$

It is also useful to calculate the derivative of A with respect to $\tilde{z}$ at constant ξ, namely $A'_{\tilde{z}}=\lim_{\Delta\tilde{z}\to0}(1/\Delta\tilde{z})\int_{S(\xi,\tilde{z}+\Delta\tilde{z})-S(\xi,\tilde{z})}\mathrm{d}S$. Figure 3 shows the superimposed areas $A(\xi,\tilde{z}+\Delta\tilde{z})$ and $A(\xi,\tilde{z})$ enclosed by the contours $\psi(x,y,\tilde{z}+\Delta\tilde{z})=\xi$ and $\psi(x,y,\tilde{z})=\xi$, respectively. Using Taylor series $\psi(x,y,\tilde{z}+\Delta\tilde{z})=\xi$ may be approximately replaced by $\psi(x,y,\tilde{z})+\Delta\tilde{z}\partial\psi/\partial\tilde{z}=\xi$. Finally, considering that the surface element between these two contours is $\mathrm{d}S=h(s)\,\mathrm{d}s$, and noticing that $h(s)$ from the inner contour to the outer contour is given by $h=-(\Delta\tilde{z}\partial\psi/\partial\tilde{z})/(\partial\psi/\partial n)$, we obtain after taking the limit

$$A'_{\tilde{z}}=-\int_C\frac{\partial\psi/\partial\tilde{z}}{\partial\psi/\partial n}\,\mathrm{d}s. \tag{3.13}$$

3.4. *Local conditions*

The extra conditions (3.9) and (3.11) obtained so far are non-local in ξ in the sense that they contain surface integrals of the type $\int_S\partial\phi(\psi,\tilde{z})/\partial\tilde{z}\,\mathrm{d}S$. With the aim of making them local, we introduce two new quantities:

$$U(\xi,\tilde{z})=-\int_S\frac{\partial W}{\partial\tilde{z}}\,\mathrm{d}S,\qquad G(\xi,\tilde{z})=-\int_S\frac{\partial\Omega}{\partial\tilde{z}}\,\mathrm{d}S. \tag{3.14}$$

Differentiating these relations with respect to ξ while keeping $\tilde{z}$ constant and taking into account that $\partial\phi(\psi,\tilde{z})/\partial\tilde{z}|_{x,y}=\phi'_\psi\partial\psi/\partial\tilde{z}+\phi'_{\tilde{z}}$, reduces (3.14) to

$$U'_\xi=W'_\xi A'_{\tilde{z}}-W'_{\tilde{z}}A'_\xi,\quad G'_\xi=\Omega'_\xi A'_{\tilde{z}}-\Omega'_{\tilde{z}}A'_\xi. \tag{3.15}$$

Once U and G have been introduced, (3.9) and (3.11) read

$$\Omega U-WG=-(1/k)\,\Omega'_\xi\Gamma, \tag{3.16}$$

$$-W'_\xi U+G=-H'_\xi A'_{\tilde{z}}+H'_{\tilde{z}}A'_\xi+(1/k)\left[W'_\xi\Gamma\right]'_\xi. \tag{3.17}$$

Observe that (3.17) reduces to Blennerhasset's result when the Bernoulli function H is linearly dependent on the axial $\tilde{z}$ direction (Blennerhassett 1979). Finally, if we apply the change of variables $(\xi,\tilde{z})\to(A,\tilde{z})$ to (3.15)–(3.17), these equations become, respectively,

$$U'_A + W'_{\tilde z} = 0, \quad G'_A + \Omega'_{\tilde z} = 0, \tag{3.18}$$

$$\Omega U - WG = -(1/k)\,\Omega'_A A'_\xi \Gamma, \tag{3.19}$$

$$-W'_A U + G/A'_\xi - H'_{\tilde z} = (1/k)\left[W'_A A'_\xi \Gamma\right]'_A. \tag{3.20}$$

It is possible to formulate certain boundary conditions for some of the dependent variables in order to simplify (3.18)–(3.20). Functions $U(A,\tilde z)$ and $G(A,\tilde z)$ may be interpreted as fluxes through the closed contour C: $\psi=\xi, \tilde z=$constant. Consequently for a null contour or null area A, these fluxes must be zero: $U(0,\tilde z)=G(0,\tilde z)=0$. The same argument applies to Γ, which depends on the contour of integration. Hence, from $\Gamma(0,\tilde z)=0$ it follows that $\Gamma(0,\tilde z)'_{\tilde z}=0$. From the definition of Γ, it follows that $\Gamma'_A-\Omega=0$. This equation combined with $G'_A+\Omega'_{\tilde z}=0$ may be rewritten as $G=\phi(\tilde z)-\Gamma'_{\tilde z}$ from which we obtain that $\phi(\tilde z)=0$ in order to satisfy the regular boundary conditions. With this result, G can be related to Γ as $G=-\Gamma'_{\tilde z}$. Substituting the latter result into (3.19)–(3.20) and considering (2.2), a closed system for the dependent variables W, U, Γ,H and Λ in the coordinates $(A,\tilde z)$ and the streamfunction ψ may be formulated as

$$U'_A + W'_{\tilde z} = 0, \tag{3.21}$$

$$\widetilde H'_A - (\Gamma/A\Lambda)\,\Gamma'_A = 0, \tag{3.22}$$

$$U\Gamma'_A + W\Gamma'_{\tilde z} = (A\Lambda/k)\,\Gamma''_{AA}, \tag{3.23}$$

$$UW'_A + WW'_{\tilde z} = -\widetilde H'_{\tilde z} + (\Gamma/A\Lambda)\,\Gamma'_{\tilde z} + (1/k)\left(A\Lambda W'_A\right)'_A, \tag{3.24}$$

$$\nabla^2_{(2)}\psi(x,y,\tilde z) = -\Gamma'_A, \tag{3.25}$$

$$\Lambda = -(\Gamma/A)\oint_{\psi=\xi}(\partial\psi/\partial n)^{-1}\,\mathrm{d}s, \tag{3.26}$$

$$A(\xi,\tilde z) = \int_{\psi(x,y,\tilde z)<\xi}\mathrm{d}x\,\mathrm{d}y, \tag{3.27}$$

where $\widetilde H = H - W^2/2$. System (3.21)–(3.27) is the main result of the present study.

3.5. *Discussion*

In the axisymmetric case, that is when the contours $\psi=$ constant are concentric circles $r=$constant, (3.26) and (3.27) reduces to $\Lambda=4\pi$, $A=\pi r^2$ and (3.21)–(3.24) reduces to the well-known quasi-cylindrical approximation equations (Revuelta, Sanchez & Linan 2004). In the general case, (3.21)–(3.24) is coupled to the rest of the system only via $\Lambda(A,\tilde z)$. As it can easily be seen, the deviation of Λ from 4π is quadratic in the magnitude of the deviation of the shape of the contours $\psi=$constant from circles, provided that the shape deviation is sufficiently smooth. Hence, in many cases this coupling can be expected to be weak. Accordingly, one can expect a parabolic behaviour of the solutions of (3.21)–(3.24) similar to the behaviour of the quasi-cylindrical approximation equations. This also suggests that (3.21)–(3.24) requires regularity conditions $U=\Gamma=0$, $\partial W/\partial A<\infty$ at $A=0$, initial conditions of the form $W=f(A)$, $\Gamma=g(A)$ at some $\tilde z=\tilde z_0$, and boundary conditions for U, Γ, and W at the boundary $A=A_{max}(\tilde z)$ of the flow domain.

On the other hand, with Γ given as a function of A, (3.25) is a Poisson equation with a nonlinear source term, and, hence, with a Dirichlet boundary condition $\psi=$ constant at the outermost closed contour it can be expected to be well posed. Hence, with these boundary conditions the whole system (3.21)–(3.27) is a well-posed problem. Note, however, that we assume $W > 0$ everywhere. If this is not true, one can expect singularities, as it is usual for equations of boundary-layer type.

The system obtained above differs from the original Prandtl–Batchelor system because in our case in the bulk of the flow the three-dimensional effects are of the same order of magnitude as the viscous effects determining the distribution of vorticity. In the vicinity of the outermost closed contour one should expect an appearance of a boundary layer, in which the viscous effects are much more pronounced. As a result (easily verifiable by a standard boundary-layer change of variables) the boundary-layer equations turn out to be two-dimensional. Hence, many well-known results about such layers apply Batchelor (1956), Squire (1956) and Wood (1957) (see also Bunyakin, Chernyshenko & Stepanov 1988 for the latest results and further references). In general, the requirement of the existence of the solution in the boundary layer surrounding the closed contour region provides the necessary boundary conditions for Γ and W. We will demostrate this by deriving the boundary condition for W at $A = A_{max}$ in the particular case that will also be a part of the illustrative example considered further, and because it appears that this particular case, however trivial it is, was not considered before.

Consider a cylindrical pipe of non-circular cross-section. Let the pipe walls move along the axial direction $\tilde{z}$ with a constant velocity $w_w = 1$. Let also a transversal velocity of the same order be imposed on the walls so that the present theory should apply. In the boundary layer near the wall we introduce at $\tilde{z} =$ constant the arclength along the wall and the normal distance to the wall multiplied by $\sqrt{Re}$ as curvilinear coordinates s and η, respectively. After the usual substitutions and taking the limit we arrive at the boundary-layer equation

$$u_s \partial w^*/\partial s + u_\eta \partial w^*/\partial \eta = \partial^2 w^*/\partial \eta^2, \tag{3.28}$$

where u_s is the velocity component along s, u_η is the $\sqrt{Re}$ times velocity component along η, and w^* is the $\tilde{z}$ velocity component in the boundary-layer region. The boundary conditions are $w^*(s, 0, \tilde{z}) = 1$ at the wall, periodicity in s, and matching $w^* \to W(A_{max}, \tilde{z})$ as $\eta \to \infty$. Multiplying (3.28) by $w^* - 1$ and integrating over the entire domain using continuity and the boundary, periodicity and matching conditions and Green's theorem gives

$$\int_0^\infty \oint \left(\frac{\partial (w^* - 1)}{\partial \eta} \right)^2 \, \mathrm{d}s \, \mathrm{d}\eta = 0. \tag{3.29}$$

Hence, $w^* = 1$ everywhere in the boundary layer, and such a solution can satisfy the matching condition only if $W(A_{max}, \tilde{z}) = 1$. This is the boundary condition for W in (3.21)–(3.27) for the special case considered.

Let now the pipe cross-section be given by $x^2/a^2 + y^2/b^2 = 1$, and the boundary condition on the transversal component of velocity is such that it is satisfied by the velocity distribution with the streamfunction $\psi_0 = -\Omega_0(x^2/a^2 + y^2/b^2 - 1)/(2/a^2 + 2/b^2)$. This streamfunction corresponds to a constant vorticity Ω_0, and, together with $W = 1$ and $\Gamma_0 = \Omega_0 A$ gives an exact solution to (3.21)–(3.27). This solution is independent of $\tilde{z}$ and satisfies, of course, the Prandtl–Batchelor theorem. Let us now perturb this solution in the inlet plane $\tilde{z} = 0$, that is, prescribe there inlet conditions of the form $W(A, 0) = 1 + \delta f(A, 0)$, $\Gamma(A, 0) = \Omega_0 A + \delta g(A, 0)$, where $\delta \ll 1$. We will seek the solution for the perturbations as functions of A and $\widetilde{z}$:

$$U=\delta U_1(A,\widetilde{z})+\dots,\quad \Gamma=\Gamma_0+\delta\Gamma_1(A,\widetilde{z})+\dots,\quad W=1+\delta W_1(A,\widetilde{z})+\dots,$$
$$\widetilde{H}=\widetilde{H}_0+\delta\widetilde{H}_1(A,\widetilde{z})+\dots,\quad \Lambda=\Lambda_0+\delta\Lambda_1(A,\widetilde{z})+\dots,$$
$$\psi=\psi_0(x,y)+\delta\psi_1(x,y,\widetilde{z})+\dots.$$

Substituting these expansions into (3.21)–(3.24), collecting terms of order δ and assuming additionally that $\Omega_0\ll 1$, one obtains the following linear system

$$\left.\begin{aligned}(U_1)'_A+(W_1)'_{\widetilde{z}}=0,\quad \left(\widetilde{H}_1\right)'_A=0,\quad (\Gamma_1)'_{\widetilde{z}}=(\Lambda_0/k)A(\Gamma_1)''_{AA},\\ (W_1)'_{\widetilde{z}}=-\left(\widetilde{H}_1\right)'_{\widetilde{z}}+(\Lambda_0/k)\left[A(W_1)'_A\right]'_A,\end{aligned}\right\}\tag{3.30}$$

Its solution should satisfy the inlet conditions $W_1=f(A,0)$, $\Gamma_1=g(A,0)$ at $\widetilde{z}=0$, the regularity conditions $W_1=\Gamma_1=0, \partial W_1/\partial A<\infty$ at $A=0$, and the boundary conditions $U_1=\Gamma_1=W_1=0$ at the wall $A=A_{max}$. The boundary condition for Γ_1 follows directly from the results of Wood (1957). Note that the functions $f(A,\widetilde{z})$ and $g(A,\widetilde{z})$ are implied to satisfy $f(A_{max},0)=g(A_{max},0)=0$ in order to avoid the appearance of additional distinguished limits. In the simple case considered the equations for the functions of A became decoupled from the Poisson equation, from which the solution inherits only the value of $\Lambda_0=2(1/a^2+1/b^2)\pi ab$. Moreover, the equation for Γ_1 is decoupled from the rest of the system. This allows to express the solution as a sum of Bessel functions, in particular,

$$\Gamma_1=\sum_{n=1}^{\infty}C_{1n}\sqrt{A}J_1\left(2\sqrt{k\sigma_n/\Lambda_0}\sqrt{A}\right)\exp(-\sigma_n\widetilde{z}),\quad \sigma_n=\Lambda_0\mu_n^2/4kA_{max},$$

$$W_1=\sum_{n=1}^{\infty}C_{2n}\left[J_0\left(2\sqrt{k\tau_n/\Lambda_0}\sqrt{A}\right)-J_0\left(2\sqrt{k\tau_n/\Lambda_0}\sqrt{A_{max}}\right)\right]\exp(-\tau_n\widetilde{z}),$$

where μ_n denotes the eigenvalue found from $J_1(\mu_n)=0$, C_{1n} and C_{2n} are constants obtained from the initial conditions and τ_n are the eigenvalues that satisfy

$$\int_0^{A_{max}}\left[J_0\left(2\sqrt{k\tau_n/\Lambda_0}\sqrt{A}\right)-J_0\left(2\sqrt{k\tau_n/\Lambda_0}\sqrt{A_{max}}\right)\right]dA=0.$$

Eigenvalues μ_n and τ_n do not coincide, that is in this case the Γ_1 decay rate differs from the decay rate of U_1 and W_1. This is the result of our additional simplifying assumption $\Omega_0\ll 1$.

It is informative to compare this solution with the axisymmetric case $a=b$ governed by a quasi-cylindrical approximation equation. The only difference in (3.30) will be in the value of Λ_0, which in turn leads to a difference in the decay rates. One can see that as the transversal motion makes the vorticty constant along the non-circular closed contours, the rate of the diffusion of vorticity and axial velocity across closed contour changes. This is the main of the new physical mechanisms described by the theory developed in the present study. In the general case this process is further complicated by the coupling of this effect with the shape of the closed contours $\psi=$constant, governed by the Poisson equation, but this effect might be expected to be more of quantitative than qualitative nature, at least in flows not deviating too far from the axisymmentric case. It is also complicated by the coupling between Γ and U and W, but this coupling also takes place in the quasi-cylindrical approximation.

In the case of a pipe of constant cross-section the parameter k is artificial, as there is actually no scale for the variation of the solution along $\widetilde{z}$ except the scale dictated by the viscosity. This would be different if the cross-section would vary along $\tilde{z}$. Then, the case of large k would correspond to three-dimensional effects dominating over viscosity, and the evolution of the flow parameters along $\tilde{z}$ would be governed purely by inviscid dynamics. The case of small k corresponds to the viscosity dominating over the three-dimensionality, and the Prandtl–Batchelor theorem would be recovered.

It remains to notice that the way Λ enters (3.21)–(3.24) makes it somewhat similar to how a variable viscosity would enter the equations of quasi-cylindrical approximation.

4. Conclusions

The system of equations obtained in this work is an extension of the well-known Prandtl–Batchelor theorem to the case when the flow parameters vary in one direction at a rate inverserly proportional to the crossflow Reynolds number. This system is similar to the well-known system of equations of a quasi-cylindrical approximation coupled with a Poisson equation for a streamfunction. The coupling occurs via the vorticity (or circulation), governed by the part of the system similar to the equations of quasi-cylindrical approximation and entering the Poisson equation as a source term, and via another function, determined from the solution of the Poisson equation and entering the part of the system similar to the equations of quasi-cylindrical approximation in a way somewhat similar to varying viscosity.

The similarity between the obtained system and the equations of quasi-cylindrical approximation establishes a connection between two seemingly unrelated phenomena.

Finally, from the viewpoint of methodology the present work is a new confirmation of the effectiveness of Batchelor's approach to elimination of uncertainty in the main term of an asymptotic expansion in cases similar to the Prandtl–Batchelor theorem.

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